Pergamon

Int. J. Solids Structures Vol. 35, Nos 28–29, pp. 3675–3694, 1998

0020-7683/98 $19.00 + .00

PII: S0020-7683(97)00230-8

FE-ANALYSIS OF A MOVING BEAM USING LAGRANGIAN MULTIPLIER METHOD

T. R. SREERAM*
School of Mechanical Engineering, University of Bath, Claverton Down, Bath BA2 7AY, U.K.

and

N. T. SIVANERI
Department of Mechanical Engineering, West Virginia University, WV 26505, U.S.A.

(*Received* 30 *November* 1996; *in revised form* 30 *June* 1997)

Abstract—Lagrangian multiplier technique is employed to develop a *h-p* version finite element model for a certain class of dynamics problems. Legendre polynomials are used as shape functions owing to its orthogonality property. Variational principle is the basis of this formulation with essential conditions applied via Lagrangian multipliers. The example considered here is a problem of a beam moving over supports. Response behavior of the beam is investigated for various longitudinal motions. The present finite element model is validated by comparing the results with previous research work. Lagrangian multiplier implementation of the problem with finite element technique, is very effective compared to other global methods such as assumed mode technique. The inconsistencies in the model presented by Lee (1993) are also pointed out.

NOMENCLATURE

a_b^L	longitudinal acceleration of the beam, m/s^2
a_i	coefficient of Legendre polynomial
A	amplitude of periodic motion, m
C_1, C_2	constant used to describe the longitudinal motion of the beam during the repositional maneuver, m
E	modulus of elasticity, N/m^2
F_x	axial force distribution, N
$[H]$	matrix of interpolation functions or shape functions for the space domain
I	moment of inertia, m^4
k	Radius of curvature, m
$[K]$	global stiffness matrix
$[K_f^e]$	element stiffness matrix—flexural
$[K_a^e]$	element stiffness matrix—axial
$[K_\lambda]$	constraint matrix of Lagrangian multipliers corresponding to displacement terms
L	length of the beam, m
l_i	length of the element, m
$[L]$	matrix of Legendre polynomials
m	total number of element degrees of freedom
$[M]$	global mass matrix
N_g	number of Gauss–Legendre points
P_i	Legendre polynomials
$\{q\}$	generalized coordinates
$\{Q\}$	global load vector
T	period, s
u	displacement in the x-direction, m
U	total strain energy
U_b	strain energy due to bending
U_f	strain energy due to axial forces
T	kinetic energy
v	displacement in the y-direction, m
v_b^L	longitudinal velocity of the beam, m/s
w	longitudinal displacement function for the support motion, m
x_s^1	location of support no. 1
x_s^2	location of support no. 2
x_i^b	distance from the tip of the beam to the beginning of ith element

* Author to whom all correspondence should be addressed.

Greek symbols

α_i	weights for the Gaussian quadrature
γ	mass per unit length, N/m
ε_x	total strain in the x-direction
λ	Lagrangian multiplier constraints for displacements
Ω	longitudinal oscillation frequency of the beam
ω	frequency, rad/s
Π_p	total potential
ξ	Gauss–Legendre points

INTRODUCTION

In displacement based finite element method for statics problems, it is necessary to apply the zero-displacement boundary conditions (also known as essential conditions) by dropping off the rows and columns from the stiffness matrix. In the case of dynamics problems the equivalent process is to drop the corresponding rows and columns from the inertia and damping matrices as well. As an alternate to dropping rows and columns corresponding to constrained displacement degrees of freedom (d.o.f.), one can use the Lagrangian multiplier method. In the literature (Cook, 1981; Reddy, 1984; Zienkiewicz and Taylor, 1989) the Lagrangian multiplier method has been used to force satisfaction of natural boundary conditions as well as essential conditions. The question of satisfying the essential conditions, by the use of Lagrangian multipliers immediately creates two distinct problems; one due to increased storage space and secondly due to the presence of rigid body modes, but this method may be the preferred one for finite element modeling of certain class of problems. The dynamic lateral response of a beam moving longitudinally relative to supports (henceforth referred to as moving beam) falls under this class. A non-moving beam is also referred in the paper which has no longitudinal motion, but only lateral vibratory motion. In this paper, non-moving beam response problem is analysed first and then the moving beam problem as the main course. Since the central example of the paper is moving beam problem, the relevant literature related to this is presented.

The vibration of elastic bodies, having time-dependent boundary conditions were first studied by Mindlin and Goodman (1950) for a classical Euler–Bernoulli beam. The Mindlin and Goodman (1950) technique can only be applied to problems where the boundary conditions are a function of time only, and not applicable to boundary conditions that are dependent on both time and space. Tabbarock *et al.* (1974) studied the dynamics of a spacecraft antennae which were treated as cantilever beams, ejected, and withdrawn by certain amount of force, applied at the root of the cantilever. In this approach, the deflection gradients of the beam were assumed to be small and the beam was assumed to be axially rigid. Adams and Manor (1981) investigated the steady-state response of an infinitely long beam moving over a rigid foundation with step discontinuity. The application of this type of problem arises when computer tapes are pulled at very high speeds along a base (Buffinton and Kane, 1985). The behavior of a uniform beam (of Euler–Bernoulli type) of finite length, moving over two bilateral supports were first studied by Buffinton and Kane (1985). Time varying partial differential equations were derived for the lateral deflection of the beam, and the solution was sought using the assumed modes method. Yuh and Young (1991) considered a different kind of dynamic problem, wherein a beam had both translational and rotational motions. For various beam motions, the response, and the effect of axial inertia was investigated. In a study by Lee (1993), the motion of a beam on two bilateral supports was analyzed, and the equations of motion was derived using Hamilton's principle. While Buffinton and Kane (1985) used a prescribed displacement for the beam motion, Lee (1993) assumed an equivalent forcing function for the beam motion, to take place (Buffinton, 1995, 1996). Not only that Lee (1993) failed to show any quantitative comparison of his results with that of Buffinton and Kane (1985) but also it will be shown that certain results presented by Lee (1993) are found to be substantially in error, for the moving beam problem. The present problem of a finite beam moving over rigid supports using finite element method with Lagrangian multiplier technique, has not been addressed by previous authors (Sreeram, 1995).

The equations of motion are derived in the section titled formulation, assuming that the beam is isotropic, axially rigid, and the effect of supports are negligible. In the numerical implementation and results section, the results are presented for various types of motions of the beam, where comparison with previous research is brought in, and errors in the work of Lee (1993) are pointed out. Conclusions are presented at the end, followed by the Appendix section.

FORMULATION

Coordinate system

Consider two fixed supports C and D at a distance d apart as shown in Fig. 1. An inertial frame (X, Y) is defined such that its origin is attached to support C, with the X axis along CD. An Euler–Bernoulli beam FG of length L moves relative to the supports in the X direction, and has a deflection $v(X)$ in the Y direction. The deflection of the beam at the points in contact with the supports C and D at a given time are zero. The horizontal motion of the beam may be specified by prescribing $X_F(t)$. Note that X_F is always negative. A moving frame (x, y) is attached to the left end F of the beam, and moves along with the beam horizontally.

The transformation between the inertial and the moving frames is given by,

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} X - X_F \\ Y \end{Bmatrix} \tag{1}$$

The axial deflection u is small compared to the lateral deflection v. More importantly, the deformation quantities u, $\dot{u}$, $\ddot{u}$ are negligible compared to the corresponding rigid body motion characteristics of the beam X_F, $\dot{X}_F$, $\ddot{X}_F$. Thus the axial motion of any point of the beam can be described by $X_F(t)$ and its derivatives. The finite element modeling of the beam is better accomplished by referring to the moving frame (x, y) as long as care is taken to include the axial-inertia effects of the moving beam. In the (x, y) frame the "motion" of the supports is described by,

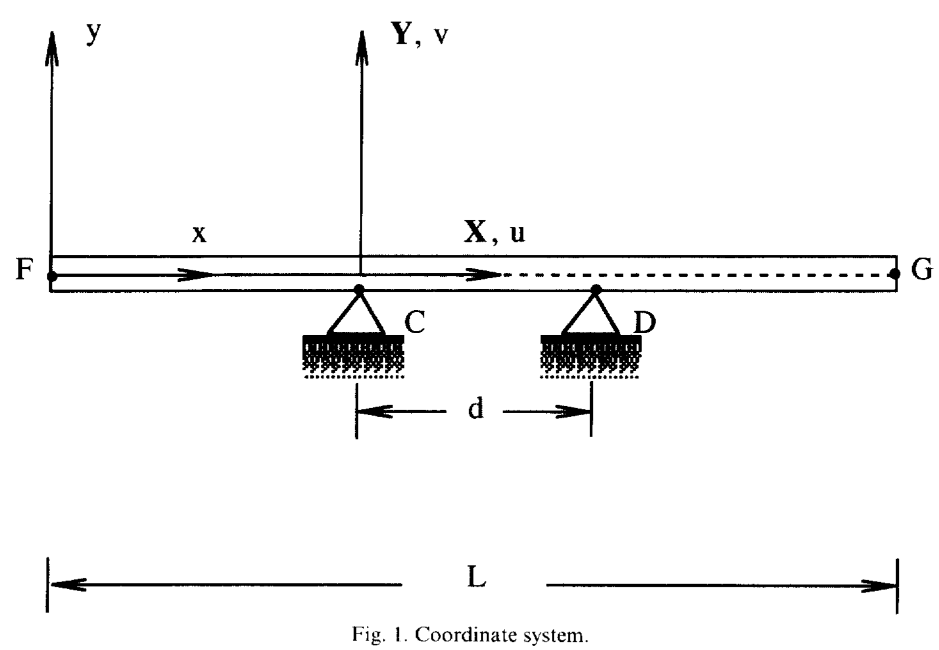

Fig. 1. Coordinate system.

$$x_C = -X_F$$
$$\text{and} \quad x_D = -X_F + d \tag{2}$$

Two types of beam motion are considered, namely oscillatory and repositioning maneuver. These motions are the same as considered by Buffinton and Kane (1985) and Lee (1993).

Oscillatory motion

The longitudinal motion of the beam is taken to be sinusoidal as,

$$X_F(t) = -w_0 + A\sin(\Omega t) \tag{3}$$

Where w_0 is the initial distance between the left end of the beam and support C; A is the amplitude and Ω the frequency of longitudinal motion of the beam. The axial velocity, v_B^L, and acceleration, a_B^L, respectively, of the beam are obtained from the eqn (1) as,

$$v_B^L = \dot{X}_F = A\Omega\cos(\Omega t)$$
$$a_B^L = \ddot{X}_F = -A\Omega^2\sin(\Omega t) \tag{4}$$

In moving coordinates, the apparent motion of the supports is given by,

$$x_C = w_0 - A\sin(\Omega t)$$
$$x_D = w_0 - A\sin(\Omega t) + d \tag{5}$$

Repositioning maneuver

This type of longitudinal motion is unidirectional, in that the beam starts from rest, moves in one direction and comes to rest. The beam motion is described by,

$$X_F = -w_0 + \frac{A}{T}\left[t - \frac{T}{2\pi}\sin\left(\frac{2\pi t}{T}\right)\right] \tag{6}$$

$$v_B^L = \dot{X}_F = \frac{A}{T}\left[1 - \cos\left(\frac{2\pi t}{T}\right)\right] \tag{7}$$

Where w_0 is as defined before, A is the distance traversed by the beam, and T the total time of the maneuver. The acceleration a_B^L during the maneuver is,

$$a_B^L = \ddot{X}_F = \frac{2\pi A}{T^2}\sin\left(\frac{2\pi t}{T}\right) \tag{8}$$

In moving coordinates, the apparent motion of the supports is given by,

$$x_C = w_0 - \frac{A}{T}\left[t - \frac{T}{2\pi}\sin\left(\frac{2\pi t}{T}\right)\right]$$
$$x_D = w_0 - \frac{A}{T}\left[t - \frac{T}{2\pi}\sin\left(\frac{2\pi t}{T}\right)\right] + d \tag{9}$$

From this point on most of the formulation will be done in the moving coordinates (x, y) excepting for the inclusion of the axial-inertia effects.

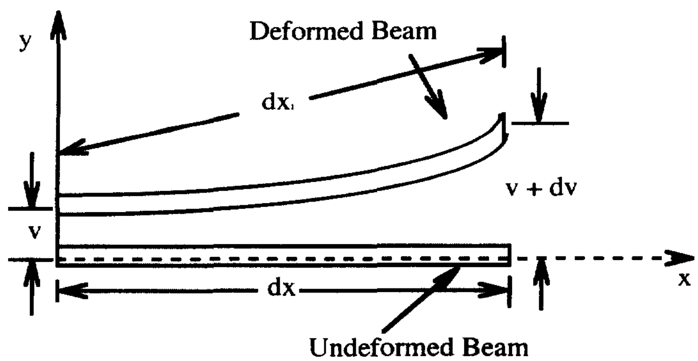

Fig. 2. Beam element.

Formation of energy equations

Figure 2 shows an element of the beam with length dx, in the undeformed state and length dx_1 in the deformed state, due to some applied load. The deflection in the axial direction is u, along the x-coordinate and in the transverse direction is v, along the y-coordinate. The equilibrium of a differential element is considered in order to derive the total potential energy due to bending and axial forces (not explicitly shown in the Fig. 2). Therefore the expressions for the axial strain components due to different actions are as follows. The total axial strain is obtained by combining the individual strains due to axial, flexural and coupling between axial and flexural actions. Therefore,

$$\varepsilon_x = \frac{\partial u}{\partial x} - y\frac{\partial^2 v}{\partial x^2} + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2 \tag{10}$$

Strain energy

The total strain energy U for the beam can be written as,

$$U = \frac{1}{2}\int_0^L \left[EA\left(\frac{\partial u}{\partial x}\right)^2 + EI\left(\frac{\partial^2 v}{\partial x^2}\right)^2 + F_x\left(\frac{\partial v}{\partial x}\right)^2\right]\mathrm{d}x \tag{11}$$

The first term is due to axial deflection is neglected. The second term is the strain energy due to bending U_b:

$$U_b = \frac{1}{2}\int_0^L \left[EI\left(\frac{\partial^2 v}{\partial x^2}\right)^2\right]\mathrm{d}x \tag{12}$$

The third term is the effect of axial forces on the transverse deflection, and denoting this term as U_f

$$U_f = \frac{1}{2}\int_0^L \left[F_x\left(\frac{\partial v}{\partial x}\right)^2\right]\mathrm{d}x \tag{13}$$

The axial force distribution F_x may be externally applied or due to inertial effects. In the current problem, F_x is due to longitudinal rigid body acceleration of the beam. It is assumed that the longitudinal motion to the beam is imparted by a time varying force applied only to the left end of the beam. Thus, the inertial force distribution will be maximum at the left end and zero at the right end, and given by (Lee, 1993).

$$F_x = -\int_x^L \gamma a_L^B \, dx \tag{14}$$

Where γ is the mass per unit length of the material. As noted earlier the acceleration is independent of x, and γ will also be independent of x for a uniform beam. Thus,

$$F_x = -\gamma a_L^B (L-x) \tag{15}$$

Kinetic energy

Any point of the beam experiences velocity in the axial and transverse directions. The axial velocity is due to the rigid body longitudinal motion and does not contribute to the equation of motion written in terms of the transverse deflection v. Thus, only considering the transverse velocity, the kinetic energy expression, T is,

$$T = \frac{1}{2}\int_0^L \gamma \left(\frac{\partial v}{\partial t}\right)^2 dx \tag{16}$$

Finite element discretization

In this section, shape functions for a p-version finite element are derived using orthogonal polynomials, i.e., Legendre polynomials. As mentioned earlier, choosing orthogonal polynomials as basis for shape functions is important in numerical integration, wherein we exploit the orthogonality property. Several researchers have favored the use of orthogonal polynomials as shape functions; for example, see Hodges (1983).

Let the beam be divided into a number of p-elements, as shown in Fig. 3. The structure of the jth element, of length l_j, is also seen in this figure. The element consists of $(m-2)$ nodes numbered from 1 to $(m-2)$. The location of the internal nodes are nothing but the zeros of the Legendre polynomial of order $(m-4)$. The total number of degrees of freedom

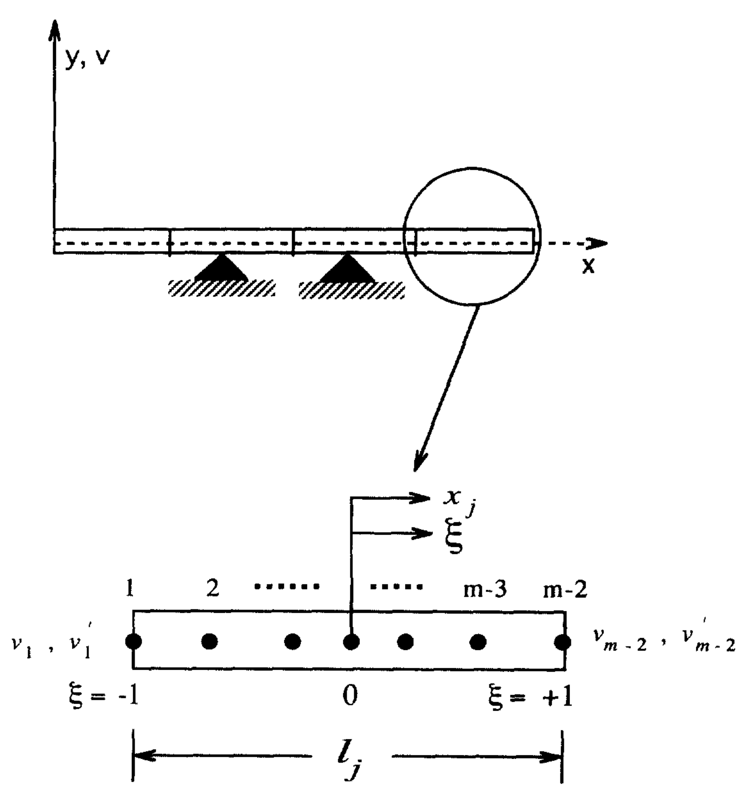

Fig. 3. p-version finite element.

for the element is m since the end nodes have the rotational degrees of freedom in addition. The local coordinate x_j and the non-dimensional coordinate ξ are both fixed at the center of the element. This origin may or may not be a nodal point. The x_j varies from $-l_j/2$ to $l_j/2$ while ξ from -1 to $+1$. The transformation between x_j and ξ is given by,

$$\xi = \frac{2x_j}{l_j} \tag{17}$$

$$\mathrm{d}\xi = \frac{2\,\mathrm{d}x_j}{l_j} \tag{18}$$

Let the deflection distribution $v(\xi)$ over the element be expressed as,

$$v(\xi) = \sum_{i=0}^{m-1} a_i P_i(\xi) \tag{19}$$

Where $P_i(\xi)$ are the Legendre polynomial of order i and a_i are the generalized coordinates which are yet to be determined. Equation (19) can be written in matrix notation as,

$$v(\xi) = \lfloor P(\xi)[\rfloor\{a\} \tag{20}$$

The solution of a_i requires m equations. The degree of freedom v and v' at each end node result in four equations, i.e.,

$$v(-1) = \lfloor P(-1)\rfloor\{a\} \tag{21}$$

$$\frac{2}{l_i}v'(-1) = \lfloor P'(-1)\rfloor\{a\} \tag{22}$$

$$v(+1) = \lfloor P(+1)\rfloor\{a\} \tag{23}$$

$$\frac{2}{l_i}v'(+1) = \lfloor P'(+1)\rfloor\{a\} \tag{24}$$

The additional $(m-4)$ equations can be found from the displacement degrees of freedom at the internal nodes as,

$$v(\xi_i) = [P(\xi_i)]\{a_i\} \quad i = 2, m-3 \tag{25}$$

Combining eqns from (20)–(25),

$$\begin{bmatrix} P_0(-1) & P_1(-1) & \dots & P_{m-1}(-1) \\ P'_0(-1) & P'_1(-1) & \dots & P'_{m-1}(-1) \\ P_0(\xi_2) & P_1(\xi_2) & \dots & P_{m-1}(\xi_2) \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ P_0(+1) & P_1(+1) & \dots & P_{m-1}(+1) \\ P'_0(+1) & P'_1(+1) & \dots & P'_{m-1}(+1) \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{m-1} \end{Bmatrix} = \begin{Bmatrix} v_1 \\ \frac{2}{l_j}v'_1 \\ v_2 \\ \vdots \\ \frac{2}{l_j}v'_{m-1} \end{Bmatrix} \tag{26}$$

or symbolically,

$$[L]\{a\} = \{q_e\} \tag{27}$$

Where $[L]$ is the square matrix of order $(m \times m)$ of eqn (27) and $\{q_e\}$ is the vector of the element degrees of freedom.

Therefore,

$$\{a\} = [L]^{-1}\{q_e\} \tag{28}$$

Combining eqns (25) and (28),

$$v(\xi) = [P(\xi)][L]^{-1}\{q_e\} \tag{29}$$

or

$$v(\xi) = \lfloor H(\xi) \rfloor \{q_e\} \tag{30}$$

The elements of the matrix $\lfloor H(\xi \rfloor$ are the shape functions of the finite element, and the shape functions are given by,

$$H_i(\xi) = [P_0(\xi) \quad P_1(\xi) \quad P_2(\xi) \quad P_3(\xi) \quad \ldots \quad P_{m-1}(\xi)][L_i]^{-1} \tag{31}$$

where $[L_i]^{-1}$ is the i^{th} column of $[L]^{-1}$.

Equations of motion

For a problem in mechanics of continua, governing equations of motion can be obtained by making use of the Hamilton's principle,

$$\delta \int_{t_i}^{t_f} (U - T - W)\,dt = 0 \tag{32}$$

The equations of motion for a moving beam will be derived based on the above approach with Lagrangian multipliers as essential conditions (displacement conditions) during the beam motion.

$$\Pi_p = [\delta U - \delta T - \delta W] \tag{33}$$

Where δU, δT and δW are the variation in strain, kinetic and virtual external work. A very detailed procedure of Lagrangian multiplier methods is given by Cook (1988). The addition of constraints to eqn (33) will yield the modified potential, i.e.,

$$\bar{\Pi}_p = [\delta U - \delta T - \delta W] + \lambda_1 v_{|x_s^1} + \lambda_2 v_{|x_s^2} \tag{34}$$

Where λ_1 and λ_2 are the two Lagrangian multiplier degrees of freedom corresponding to the two support locations, and $x_s^1(C)$, $x_s^2(D)$ are the location of the first and second support, respectively. To derive the system equations, the potential function is made stationary with respect to each degree of freedom and assembled in matrix form. In other words, mathematically this is accomplished by computing the partial derivatives of the modified potential with respect to each degree of freedom and equating them to zero.

$$\frac{\partial \Pi_p}{\partial q_1} = 0$$

$$\frac{\partial \Pi_p}{\partial q_2} = 0$$

$$\vdots$$

$$\frac{\partial \Pi_p}{\partial \lambda_1} = 0$$

$$\frac{\partial \Pi_p}{\partial \lambda_2} = 0 \tag{35}$$

Where the q's are the real degrees of freedom of the system whereas λ's are the Lagrangian Multiplier degrees of freedom. Equation (35) in the matrix form gives the system matrix equations for the moving beam problem, assuming effects due to damping are neglected.

$$\begin{bmatrix} [M] & [0] \\ [0] & [0] \end{bmatrix} \begin{Bmatrix} \ddot{q} \\ \ddot{\lambda} \end{Bmatrix} + \begin{bmatrix} [K] & [K_\lambda] \\ [K_\lambda^T] & [0] \end{bmatrix} \begin{Bmatrix} q \\ \lambda \end{Bmatrix} = \begin{Bmatrix} \{Q\} \\ \{0\} \end{Bmatrix} \tag{36}$$

Now, $[K] = [K_f] + [K_a]$, the sum of flexural and axial inertia stiffness. The expression given in eqn (36) is the general governing equation of a beam moving over supports irrespective of the type of motion. If the motion is sinusoidal, then the additional stiffness term does exist and in this case the stiffness will be due to both bending and axial. But if the beam moves with constant velocity, the stiffness term will be due to bending only. Also, $\{Q\}$, is the load vector which is problem dependent. The response is analyzed based on the initially deformed beam. The $[K_\lambda]$ and $[K_\lambda]^T$ are the Lagrangian multiplier matrices. The constraint matrices are not constant and their values change with time, as they depend on the location of the supports. The flexural stiffness, axial stiffness, Lagrangian multiplier and the mass matrices are defined below.

(a) Flexural stiffness

$$[K_f^e] = \frac{8}{(l_j)^3} \int_{-1}^{+1} EI \lfloor H''(\xi) \rfloor \{H''(\xi)\}\, \mathrm{d}\xi \tag{37}$$

(b) Axial stiffness

$$[K_a^e] = \frac{-2a_B^L}{l_j} \int_{-1}^{+1} \left[\gamma \left\{ L - \left[x_b + \frac{l_j}{2}(1+\xi) \right] \right\} \{H'(\xi)\} \lfloor H'(\xi) \rfloor \right] \mathrm{d}\xi \tag{38}$$

(c) Lagrangian multiplier matrices

The Lagrangian multiplier matrices, or the constraint matrices are evaluated based on the support position. By doing so, from eqn (35) the constraint terms separate as,

$$[K_\lambda^T] = \begin{bmatrix} 0 & H_1^i & \cdots & \cdots & H_m^i & 0 & 0 & 0 & 0 & 0 \\ & \cdots & & \cdots & & \cdots & & \cdots & & \cdots \\ 0 & 0 & 0 & 0 & 0 & H_1^j & \cdots & \cdots & H_m^j & 0 \end{bmatrix} \tag{39}$$

Where the H's are the shape function of ith and jth element, evaluated at the location of the two supports, assuming that the support falls on these elements. This need not be always

Table 1. Beam characteristics

Length of the beam (L)	1.0 m
Mass per unit length of the beam (γ)	1.0 Kg/m
Beam stiffness (EI)	1.0 N-m^2
Distance between the supports (D)	0.25 m

Table 2. Convergence study and comparison for the free-vibration analysis of an overhang beam using Lagrangian multiplier method

Mode no.	Classical solution rad/sec	Buffinton and Kane (1985)	1EL–0IN	2EL–0IN	3EL–0IN	4EL–0IN	4EL–1IN	4EL–2IN	4EL–3IN	4EL–4IN	4EL–5IN
1	16.246	16.247	19.861	16.333	16.286	16.296	16.248	16.248	16.246	16.246	16.246
2	20.771	20.825	40.141	23.518	20.965	21.714	20.826	20.826	20.784	20.784	20.776
3	117.93	117.95		170.51	130.48	119.48	118.39	117.99	117.94	117.94	117.93
4	136.07	136.95		206.41	158.11	153.97	137.61	137.08	136.29	136.29	136.16
5	247.47	248.85	—	—	468.51	313.03	255.49	252.471	248.44	248.43	247.83
6	386.11	388.79	—	—	534.87	517.28	399.95	392.32	386.90	386.84	386.37
7	422.58	427.03	—	—	—	840.11	447.23	446.19	425.92	425.89	423.74
8	702.44	707.16				993.36	906.65	719.43	708.78	704.75	703.88
9	799.47	807.16	—	—	—	—	1103.8	856.04	813.52	806.29	800.52

the case. Depending on the element size, the distance between the supports, the supports may or may not be on the same element. In either case, the matrix should be arranged accordingly.

(d) Mass matrix

$$[M^e] = \frac{l_i}{2}\int_{-1}^{+1} [\gamma\{H(\xi)\}\lfloor H(\xi)\rfloor]\,\mathrm{d}\xi \tag{40}$$

The derivation of flexural and axial stiffness matrices are shown in Appendix A, to demonstrate the procedure adopted here.

SIMULATION AND RESULTS

A computer program in FORTRAN 77, was written to form the finite element equations that govern the problem, and to solve the equations numerically. A convergence study was carried out during the early stages to determine how many degrees of freedom are required for the solution to be reasonably accurate; and some original results are presented. The assumed parameters for the beam are the same as assumed by Buffinton and Kane (1985), Lee (1993). Accordingly, the free-vibration problem of an overhang beam is considered first. Then the response of the non-moving beam and moving beam are presented.

Free vibration of an overhang beam using Lagrangian multiplier method

An overhang beam, as shown in Fig. 1 with the properties given in Table 1 was analyzed for eigenvalues. The IMSL program DGVCRG was used to solve the generalized eigenproblem. The comparison of results with classical theory and that presented by Buffinton and Kane (1985) is shown in Table 2. The results show a very good agreement with the classical theory, especially for the case with 4-elements and 5-internal nodes per element (4EL–5IN) case, with total beam degree of freedom of 32, where the first nine mode frequencies were obtained within reasonable accuracy. This was fixed as the basis for obtaining the required degrees of freedom, to obtain numerical solutions.

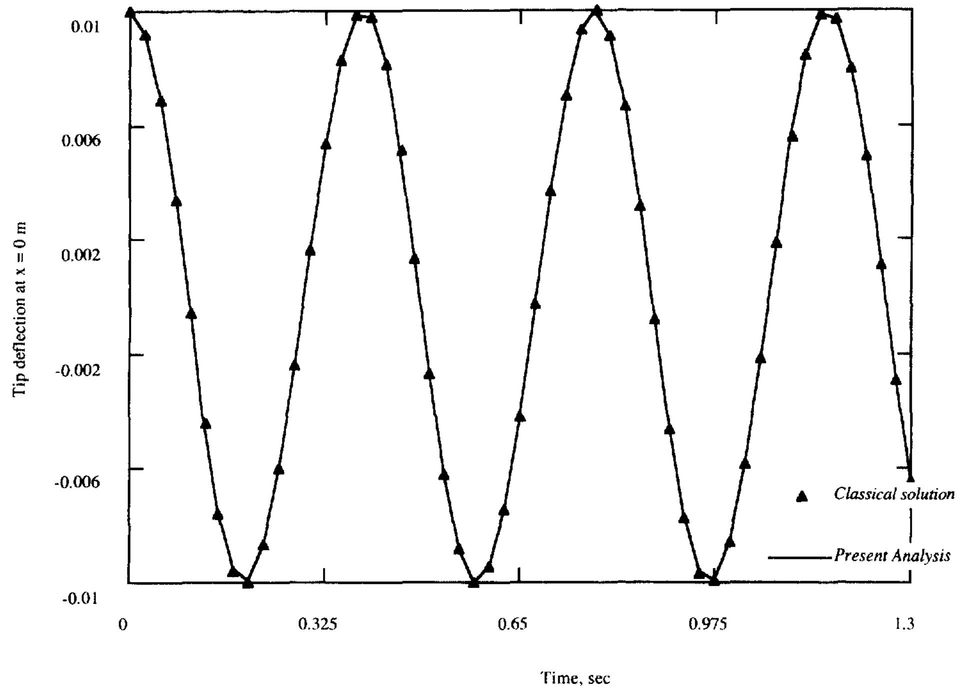

Fig. 4. Response of an initially deformed beam : comparison with classical theory.

Response analysis of an overhang beam

The response of an overhang non-moving beam is presented in this section. The analysis was carried out by using the Newmark's implicit method for solving second-order differential equations (Bathe and Wilson, 1977). The results are presented for two cases: one, assuming that the beam is deformed to the first flexural mode shape; and second, the beam is deformed by a uniformly applied load. In both cases the deflection at the tip is 0.01 m. The result of the response is as shown in Fig. 4.

As can be seen from Fig. 4, the numerical solution obtained using Newmark's method agrees with the classical solution. More importantly, the response obtained using the two different initial conditions, one using the eigenvector and the other using deflection curve due to uniformly distributed load (UDL), agree very well. However, in the case of moving beam problem, only the initial condition due to the first mode eigenvector will be considered, though one could choose either of the two as stated by Buffinton and Kane (1985).

Response of moving beam

Sinusoidal motion

In the case of a beam with longitudinal motion over supports, the axial inertia of the beam alters the stiffness matrix in the finite element analysis. The total degrees of freedom required for convergence was assumed to be four elements and five internal nodes per element. The dynamic behavior of the beam is presented for various longitudinal motions. As stated before, the beam characteristics are the same as given by Buffinton and Kane (1985) and Lee (1993). The first support is located at 0.375 m from the left and the second support is 0.25 m from the first support. The beam is considered to perform sinusoidal oscillation about the first support, and hence the longitudinal function which describes the location of supports at any time 't' is given by eqn (5)

As the location of support changes with every instant of time, this changes the Lagrangian multiplier matrices, i.e., the constraint matrices as well as with the stiffness terms. So, at each instant of time, the overall stiffness matrix will have to be updated for the constraint terms as well as the stiffness terms. Though a detailed numerical stability behavior

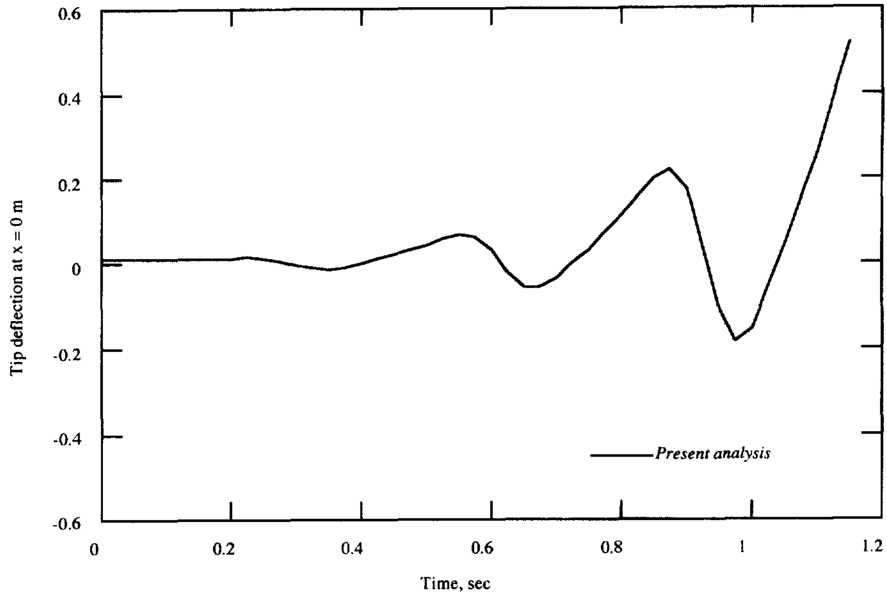

Fig. 5. Response of a moving beam: w = 0.375–0.05 sin(20 t), present analysis.

will not be presented in this paper, present analysis showed the Newmark's implicit scheme to be the most stable, and hence will be followed throughout our analysis.

Figure 5 shows the tip deflection (at $x = 0$ m) for the beam oscillating with longitudinal frequency of 20 rad/sec and amplitude of 0.05 m. Here one must remember that Buffinton and Kane (1985) used only two modes for this particular analysis. In evaluating the response, choosing the number of modes becomes critical because this will more accurately define the initial mode shape. Lee (1993) has shown the effect of number of modes on the response in his work which is a good indication that a minimum of four modes will have to be considered to define the problem accurately. The comparison is excellent in the case of repositioning maneuver of the beam, shown later, where Buffinton and Kane used five modes instead of two.

The same problem with exactly the same parameters was also presented by Lee (1993), where the two results differ considerably. This is due to the following reason. The expression for x_c eqn (5), describes the location of the supports and not of the beam. Differentiating eqn (5) with respect to time,

$$\ddot{x}_C = A\Omega^2 \sin(\Omega t) \tag{41}$$

The expression in eqn (41) is just the second derivative of the expression for x_c, and not the beam acceleration. The actual acceleration of the beam is defined by eqn (4). Lee (1993) has not used the minus sign for the term in eqn (41). Also it will be shown that by using the opposite sign for the acceleration term, the results show a very good agreement with that of Lee (1993). The results presented by Lee (1993) for the same case with support stiffness of 1.0×10^7 N/m, were digitized for the sake of comparison. Figure 6 represents the response case for the beam oscillating with longitudinal frequency of 20 rad/s and amplitude of 0.05 m, but with an opposite sign for the acceleration time. The results for higher frequencies is presented due to the fact that the effect of axial inertia will be more pronounced, and it drastically changes the stiffness terms and hence the response behavior.

Next the results are presented for a case where the longitudinal frequency of the beam is 20 rad/s, but the amplitude is lowered to 0.025 m, as shown in Fig. 7. The results presented by Lee (1993) shows a very stable behavior, whereas the present analysis shows instability progressing with increase in time (Fig. 7). The response behavior is shown only for five seconds. At this point it should be noted that the stability chart presented for the same

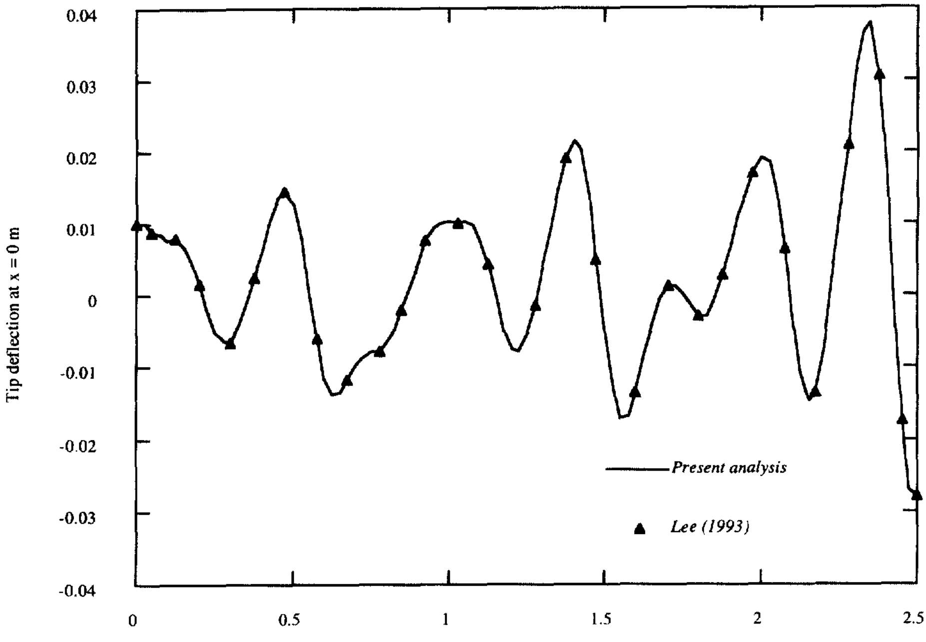

Fig. 6. Response as a moving beam: $w = 0.375–0.05\sin(20\ t)$, present analysis with opposite sign for beam acceleration.

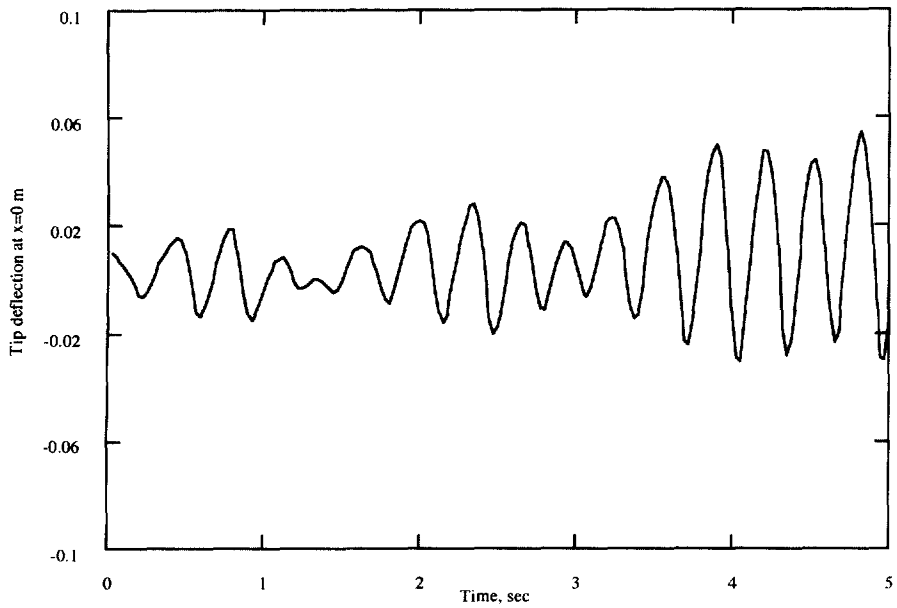

Fig. 7. Response of a moving beam: $w = 0.375–0.025\sin(20\ t)$, present analysis.

problem by Buffinton and Kane (1985), is only applicable to discrete values of amplitude and longitudinal frequency, which, perhaps, Lee (1993) did not realize since he brings in a wrong comparison in his paper. This essentially means that one cannot even qualitatively compare, or predict the response for an amplitude of 0.025 m and frequency of 20 rad/s from the stability chart provided by Buffinton and Kane (1985). The response result for this particular case was further investigated using Floquet's theory (results not presented here), also yielded unstable results. The response of beam having longitudinal frequency of 22 rad/s with amplitude 0.05 m is presented next, which clearly shows unstable behavior in

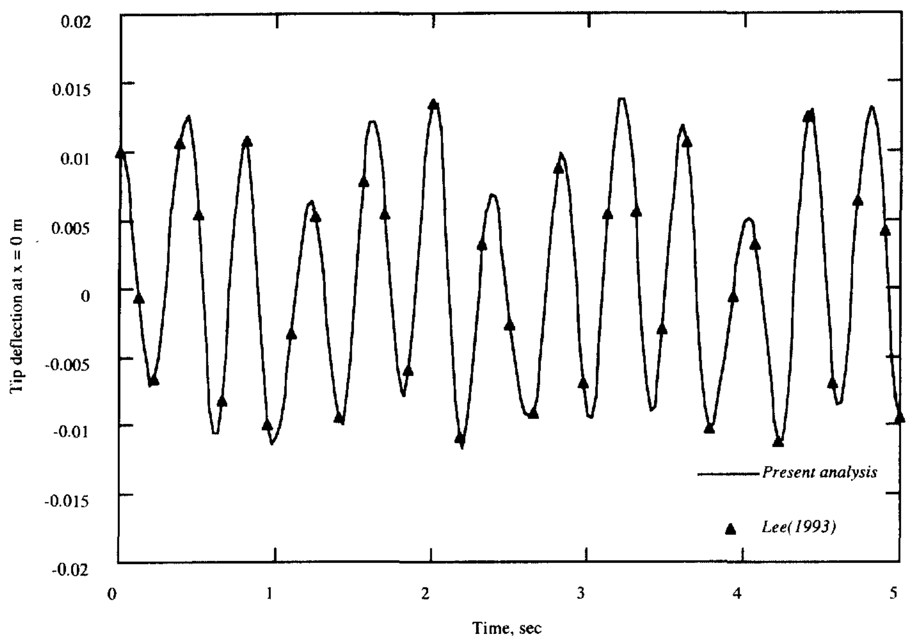

Fig. 8. Response of a moving beam: $w = 0.375\text{–}0.025\sin(20\,t)$, present analysis with opposite sign for beam acceleration.

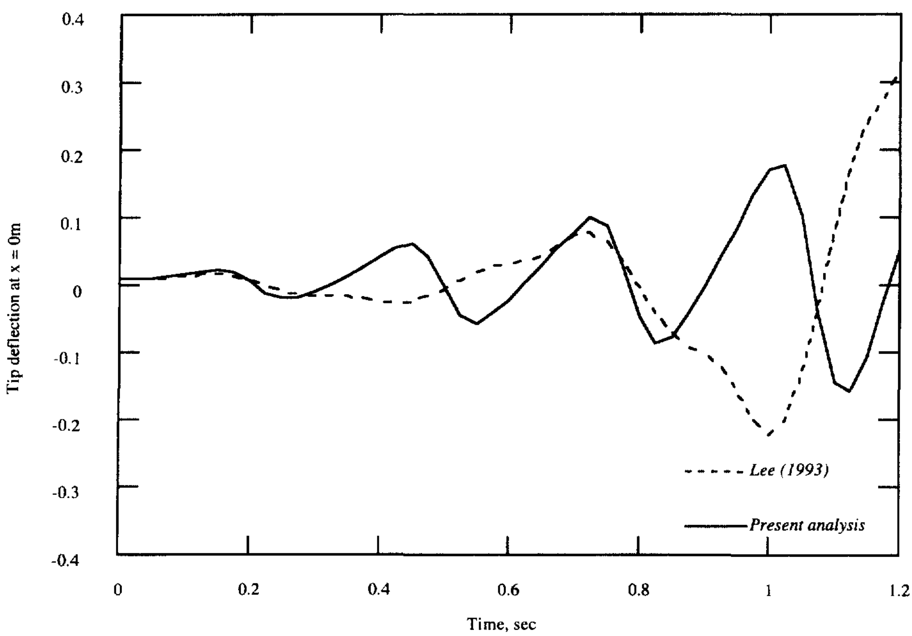

Fig. 9. Response of a moving beam: $w = 0.375\text{–}0.05\sin(22\,t)$, present analysis showing the effect of sign change for the beam acceleration.

the present analysis. Figure 9 shows the results obtained using our analysis which again differs quantitatively with that presented by Lee (1993) due to the sign change in the acceleration term. The analysis was also performed for other longitudinal frequencies of 30 and 40 rad/s keeping the amplitude the same, which also yielded unstable responses as can be seen from Figs 10 and 11. The results for these two cases were not presented by previous authors.

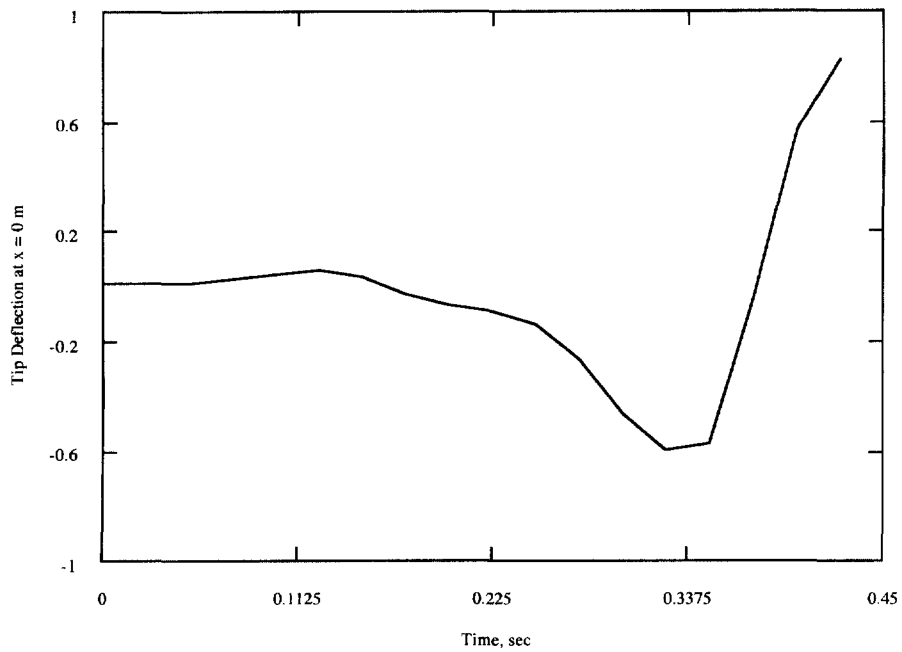

Fig. 10. Response of a moving beam: $w = 0.375\text{–}0.05\sin(30\ t)$, present analysis.

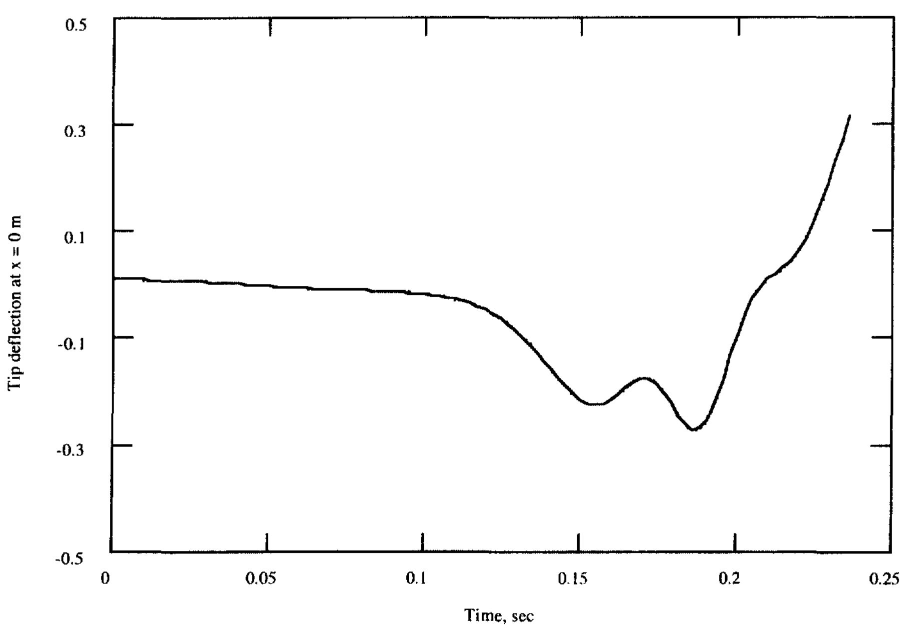

Fig. 11. Response of a moving beam: $w = 0.375\text{–}0.05\sin(40\ t)$, present analysis.

Repositional maneuver

The results for the repositional maneuvers are presented next. The functions that describe the beam motions, in eqn (9) are the same as given by Buffinton and Kane (1985) and Lee (1993). Figure 12 shows the case of slow repositioning maneuver. The tip deflection of the beam is observed up to 3.5 s, and one can observe the decrease in amplitude as the tip approaches the first support. The predicted results agree very well with that of Buffinton and Kane (1985) for all three cases of repositioning. Figure 15 shows the results for the slow repositional maneuver of the beam based on the initial configuration given by Lee

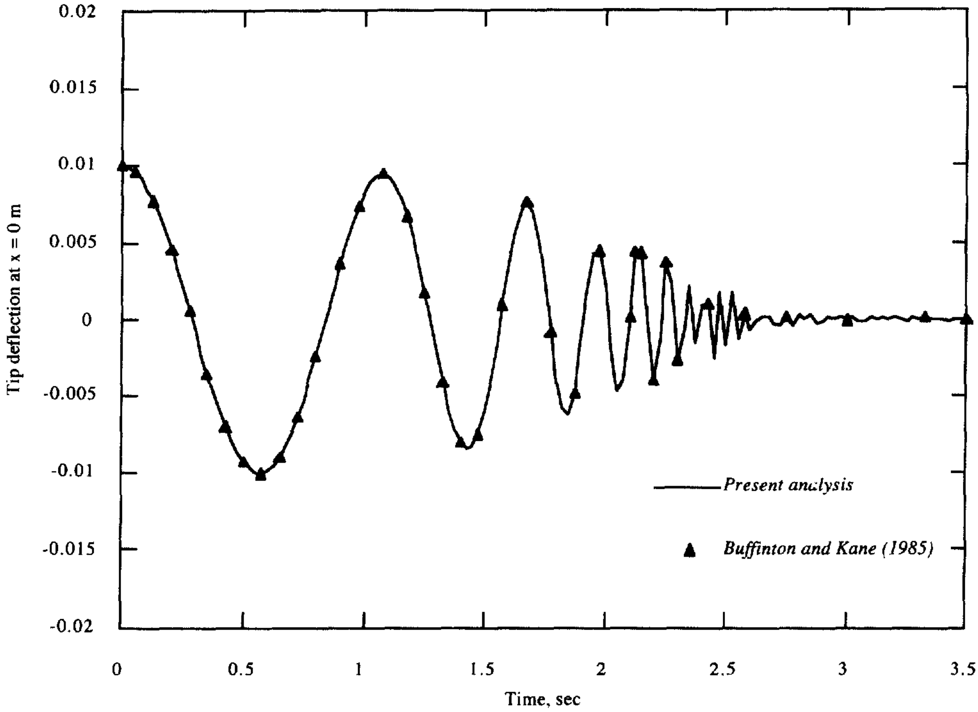

Fig. 12. Response of a beam undergoing slow repositional maneuver: $w = C_1-(C_2/T)[t-(T/2\pi)\sin(2\pi t/T)]$, $C_1 = 0.725$ m, $C_2 = 0.700$ m and $T = 3.50$ s, comparison with Buffinton and Kane (1985).

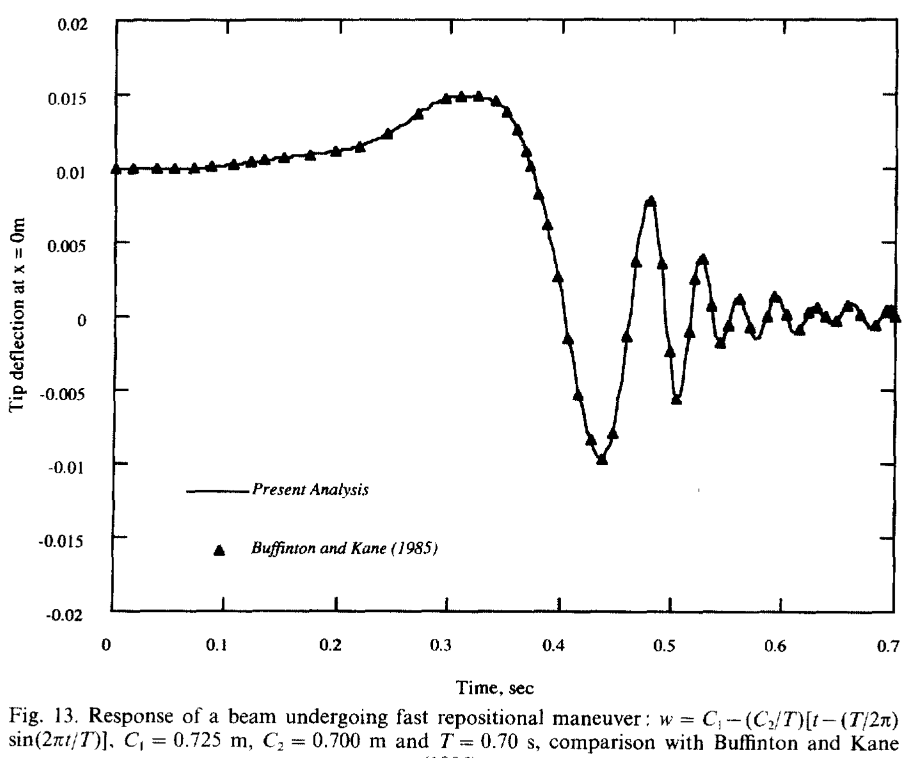

Fig. 13. Response of a beam undergoing fast repositional maneuver: $w = C_1-(C_2/T)[t-(T/2\pi)\sin(2\pi t/T)]$, $C_1 = 0.725$ m, $C_2 = 0.700$ m and $T = 0.70$ s, comparison with Buffinton and Kane (1985).

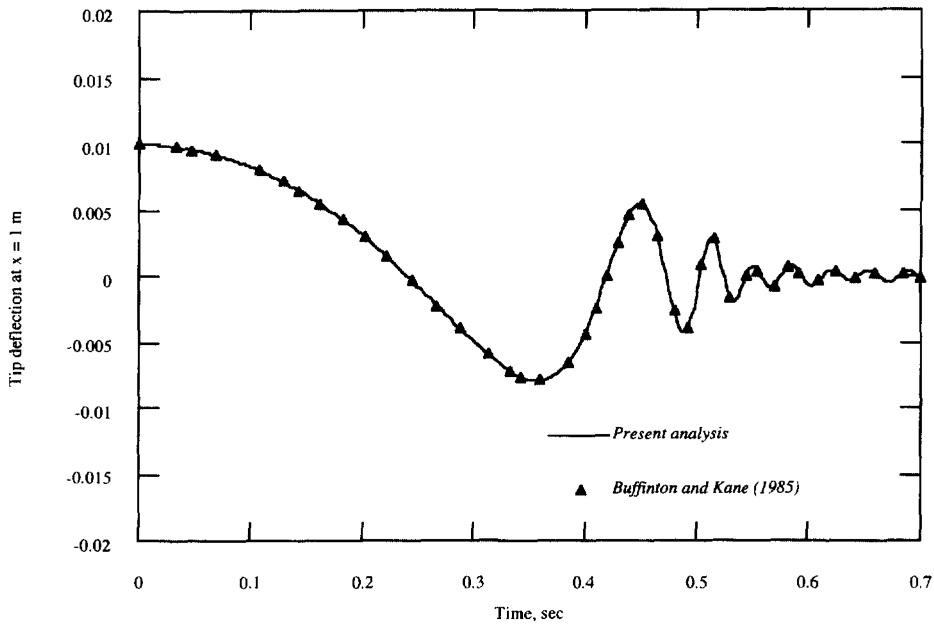

Fig. 14. Response of a beam undergoing rapid reverse respositional maneuver: $w = C_1-(C_2/T)[t-(T/2\pi)\sin(2\pi t/T)]$, $C_1 = 0.025$ m, $C_2 = -0.700$ m and $T = 0.70$ s, comparison with Buffinton and Kane (1985).

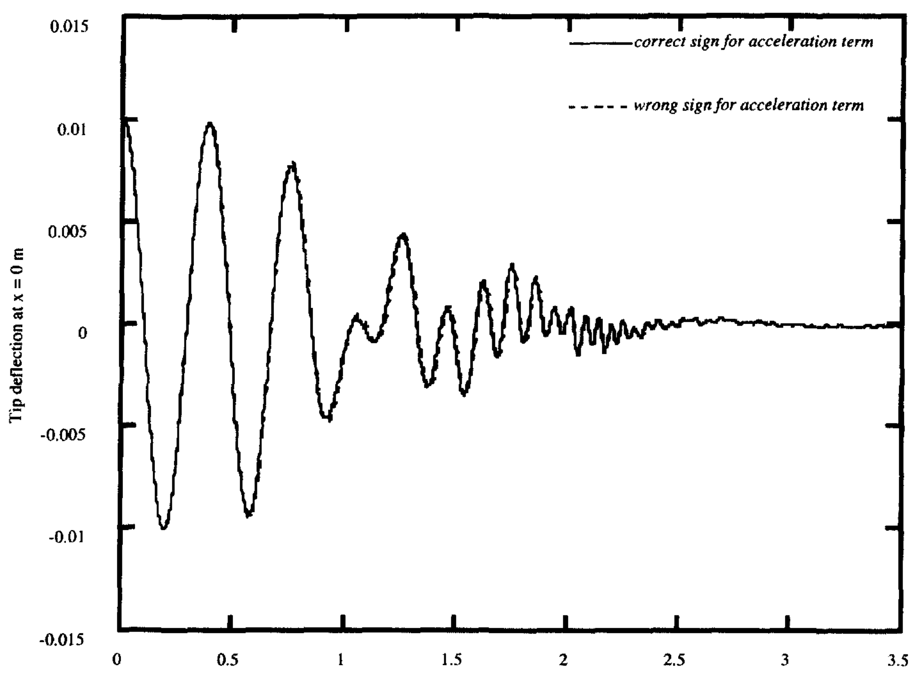

Fig. 15. Response of a beam undergoing slow repositional maneuver: $w = C_1-(C_2/T)[t-(T/2\pi)\sin(2\pi t/T)]$, $C_1 = 0.375$ m, $C_2 = 0.350$ m and $T = 3.50$ s, present analysis showing the effect of sign change for the beam acceleration.

(1993). Figure 15 shows the results obtained using the reversal of signs in the acceleration terms. For this particular case, the effect seems to be very small, since the beam motion is slow. Similar results are also presented for a fast repositional maneuver of the beam, initial

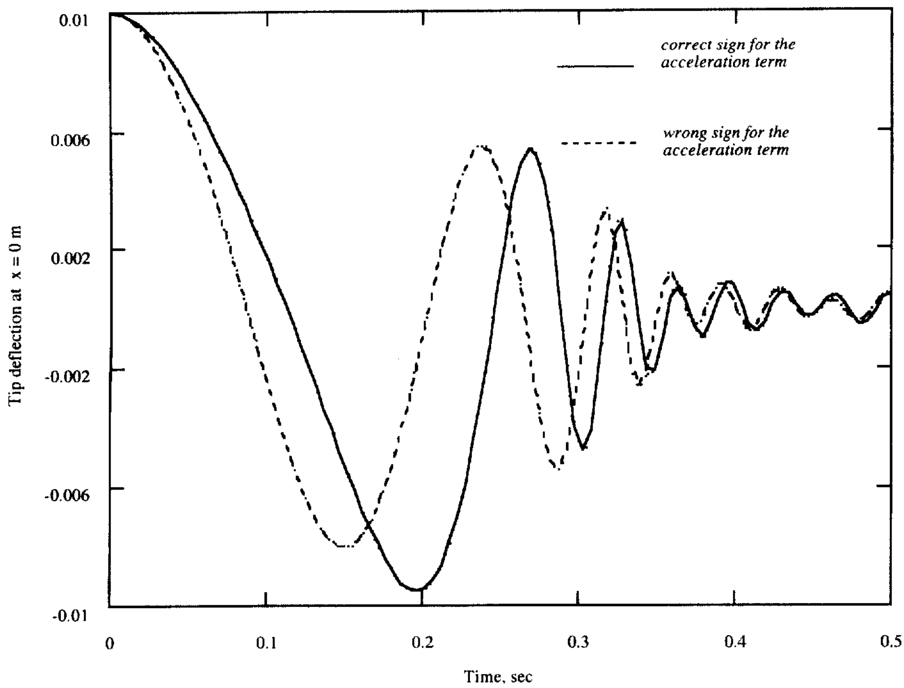

Fig. 16. Response of a beam undergoing fast repositional maneuver: $w = C_1-(C_2/T)[t-(T/2\pi)\sin(2\pi t/T)]$, $C_1 = 0.375$ m, $C_2 = 0.350$ m and $T = 0.50$ s, present analysis showing the effect of sign change for the beam acceleration.

configuration given by Lee (1993), where the effect of sign reversal is very pronounced, as can be seen from Fig. 16.

CONCLUSIONS

The *h-p* version finite element method in combination with Lagrangian multiplier technique, has proved to be very effective in solving problems with time-dependent boundary conditions. The formulation presented here is much less complex than the assumed modes solution presented by Buffinton and Kane (1985) and Lee (1993).

The use of Legendre polynomials as shape functions have many advantages due to its orthogonality property. The Gauss–Legendre points are nothing but the zeros of the Legendre polynomial of a certain order. This is especially advantageous when one has to generate the internal nodes. For the response of a moving beam, the results show excellent comparisons with that of Buffinton and Kane (1985). The erroneous results of the dynamic model presented by Lee (1993) are pointed out, and the corrected model using a much simpler formulation is presented for the same problem.

Acknowledgement—The first author would like to express his sincere thanks to the Brahmin Educational Society and Sanskrit Academy, Palghat, India, for their much needed support during the course of this work. The first author wishes to express his heartfelt thanks to his parents, Mrs Santha Ramani and Mr Ramani Iyer, whose understanding and encouragement were a constant consolation during the long arduous process of completing this work.

REFERENCES

Adams, G. G. and Manor, H. (1981) Steady motion of an elastic beam across a rigid step. *Transactions of the ASME, Journal of Applied Mechanics* **48**, 606–612.

Bathe, K. J. and Wilson, E. L. (1976) *Numerical Methods in Finite Element Analysis*. Prentice-Hall Inc., Englewood Cliffs, New Jersey.

Buffinton, K. W. (1995) The effect of driving force application point on the dynamics and stability of a beam moving over supports. *Proceedings of the Fourth Pan American Congress of Applied Mechanics*, pp. 1031–1040.

Buffinton, K. W. (1996) Private Communication, Department of Mechanical Engineering, Bucknell University, Lewisburg, PA 17838, U.S.A.

Buffinton, K. W. and Kane, T. R. (1985) Dynamics of a beam moving over supports. *International Journal of Solids and Structures* **21**(7), 617–643.

Cook, R. D. (1981) *Concepts and Applications of Finite Element Analysis*, 2nd edn. John Wiley and Sons Inc., New York.

Hodges, D. H. (1983) Orthogonal polynomial as variable-order finite element shape functions. *AIAA Journal* **21**(5), 796–797.

Lee, H. P. (1993) Dynamics of beam moving over multiple supports. *International Journal of Solids and Structures* **30**(2), 199–209.

Mindlin, R. D. and Goodman, L. E. (1950) Beam vibrations with time-dependent boundary conditions. *Transactions of the ASME, Journal of Applied Mechanics* **17,** 377–380.

Reddy, J. N. (1984) *Energy and Variational Methods in Applied Mechanics*. John Wiley and Sons, New York.

Sreeram, T. R. (1995) Dynamics of a moving beam using *h-p* version finite element method with essential conditions applied via Lagrange multipliers. M.S. thesis, Department of Mechanical and Aerospace Engineering, West Virginia University, Morgantown, West Virginia, 26506.

Tabarrock, B., Leech, C. M. and Kim, Y. (1974) On the dynamics of an axially moving beam. *Journal of Franklin Institute* **297**(3), 000–000.

Yuh, J. and Young, T. (1991) Dynamic modeling of an axially moving beam in rotation: simulation and experiment. *Journal of Dynamic Systems, Measurement and Control* **113,** 34–40.

Zienkiewicz, O. C. and Taylor, R. L. (1989) *The Finite Element Method*, 4th edn, Vol. 1. McGraw-Hill, New York.

APPENDIX

Formation of element stiffness matrix form elastic strain energy due to bending

From eqn (12), the expression for the elastic strain energy due to bending over the jth element can be written as,

$$U_b = \frac{1}{2}\int_{-(l_j/2)}^{+(l_j/2)} \left[EI \left(\frac{\partial^2 v}{\partial x_j^2} \right)^2 \right] \mathrm{d}x_j \tag{A1}$$

where

$$\{v\} = \lfloor H(\xi) \rfloor \{q_e\} \tag{A2}$$

$$\delta U_b = \int_{-(l_j/2)}^{+(l_j/2)} \left[EI \left(\frac{\partial^2 v}{\partial x_j^2} \right) \left(\frac{\partial^2 \delta v}{\partial x_j^2} \right) \right] \mathrm{d}x_j \tag{A3}$$

From the expression for v in eqn (A2), the variation δv is,

$$\{\delta v\} = \lfloor H(\xi) \rfloor \{\delta q_e\} \tag{A4}$$

Substituting eqns (A2) and (A4) in eqn (A3) and simplifying yields the expression for the element stiffness matrix in terms of the local coordinate x_j,

$$[K_f^e] = \int_{-(l_j/2)}^{+(l_j/2)} EI \lfloor H'' \rfloor \{H''\} \, \mathrm{d}x_j \tag{A5}$$

In the eqn (A5), the integration is done with respect to the local coordinate x. Now we need to express the same equation in terms of the non-dimensional ξ coordinate, which varies from -1 to $+1$ over an element (as shown in Fig. 3).

$$H'(x_j) = \frac{\mathrm{d}H}{\mathrm{d}\xi}\frac{\mathrm{d}\xi}{\mathrm{d}x_j} \tag{A6}$$

or

$$H'(x_j) = \frac{2}{l_j} H'(\xi) \tag{A7}$$

$$H''(x_j) = \left(\frac{2}{l_j} \right)^2 H''(\xi) \tag{A8}$$

Substituting eqn (A8) in (A5) and on simplification yields,

$$[K_f^e] = \frac{8}{(l_j)^3} \int_{-1}^{+1} EI \lfloor H''(\xi) \rfloor \{H''(\xi)\} \, \mathrm{d}\xi \tag{A9}$$

Applying the Gauss integration to eqn (9A),

$$[K_f^e] = \frac{8}{(l_j)^3} \sum_{i=0}^{N_g} \alpha_i EI \lfloor H''(\xi) \rfloor \{H''(\xi)\} \tag{A10}$$

Formation of axial stiffness matrix from strain energy due to axial forces

The strain energy due to axial forces corresponding to the jth element can be written from eqn (13) as,

$$U_{f_j} = \frac{1}{2} \int_{-(l_j/2)}^{+(l_j/2)} \left[F_x \left(\frac{\partial v}{\partial x_j} \right)^2 \right] \mathrm{d}x_j \tag{A11}$$

Where F_x is the axial force corresponding to that particular element. The expression for the axial force term is given by eqn (15). Substituting eqn (15) in eqn (A11),

$$U_{f_j} = \frac{-1}{2} \int_{-(l_j/2)}^{+(l_j/2)} \left[\gamma a_B^L (L - x_j) \left(\frac{\partial v}{\partial x_j} \right)^2 \right] \mathrm{d}x_j \tag{A12}$$

The variation of eqn (A12) with respect to v yields,

$$\delta U_{f_j} = -a_B^L \int_{-(l_j/2)}^{+(l_j/2)} \left[\gamma (L - x) \left(\frac{\partial v}{\partial x_j} \right) \left(\frac{\partial \delta v}{\partial x_j} \right) \right] \mathrm{d}x_j \tag{A13}$$

Further using eqns (A2) and (A4),

$$\delta U_{f_j} = -a_B^L \int_{-(l_j/2)}^{l_j/2} [\gamma (L - x) \lfloor \delta q \rfloor \{H'\} \lfloor H' \rfloor \{q\}] \, \mathrm{d}x_j \tag{A14}$$

Transformation to the non-dimensional coordinate ξ results in,

$$\delta U_{f_j} = \frac{-2a_B^L}{l_j} \int_{-1}^{+1} \left[\gamma \left\{ L - \left[x_b + \frac{l_j}{2}(1+\xi) \right] \right\} \{H'(\xi)\} \lfloor H'(\xi) \rfloor \right] \mathrm{d}\xi \tag{A15}$$

Applying Gauss integration,

$$[K_a^e] = \frac{-2a_B^L}{l_j} \sum_{i=1}^{N_g} \left[a_i \gamma \left\{ L - \left[x_b^j + \frac{l_j^i}{2}(1+\xi) \right] \right\} \{H'(\xi)\} \lfloor H'(\xi) \rfloor \right] \tag{A16}$$

The derivation for mass matrix and load vector will not be shown here since the procedure is very similar in principle.